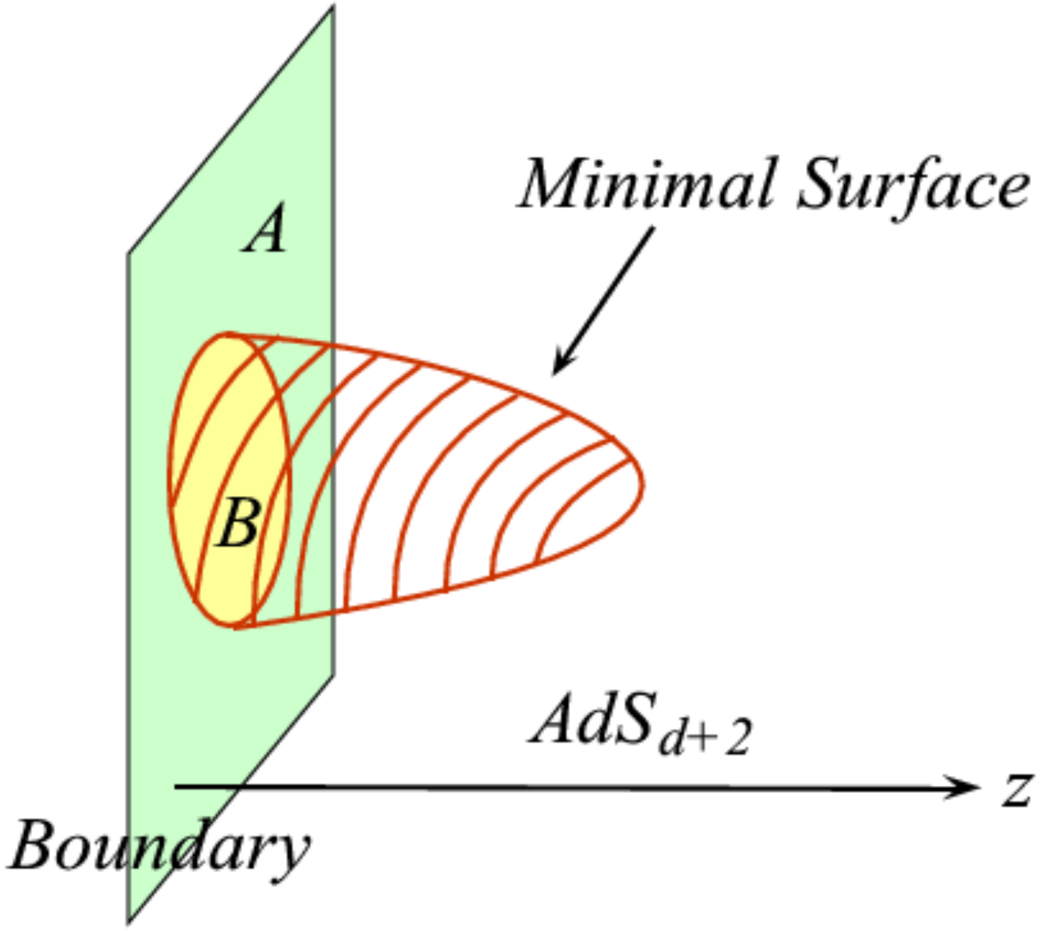
The connection between holographic entanglement and complexity of purification

arXiv:1902.02475

Mahdis Ghodrati

Wuhan 2019

Holographic Entanglement Entropy



$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

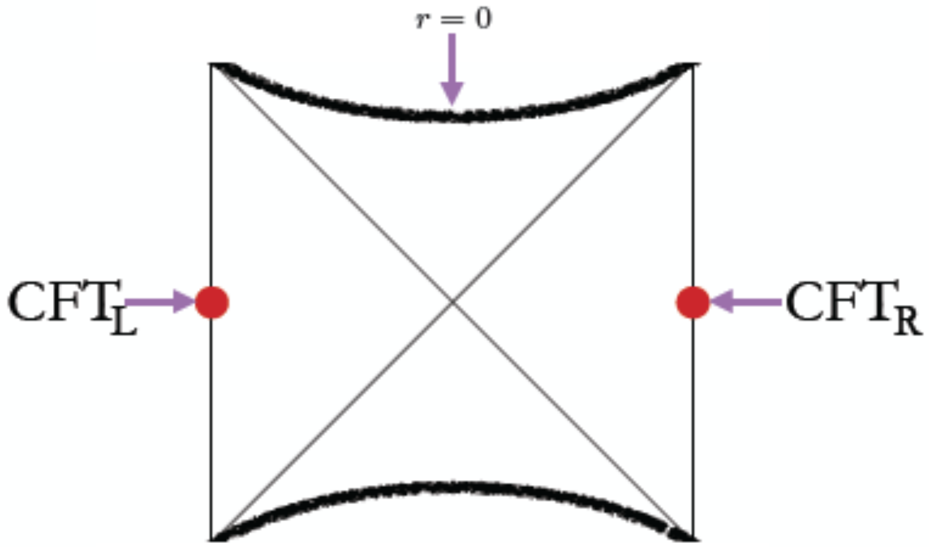
$$\rho_A = \mathrm{Tr}_{A^c} \rho_{tot}$$

$$S_A := -tr \rho_A \log \rho_A$$

Entanglement structure contains geometric data!

Ryu & Takayanagi 2006

What is the CFT dual to linear growth of wormhole?



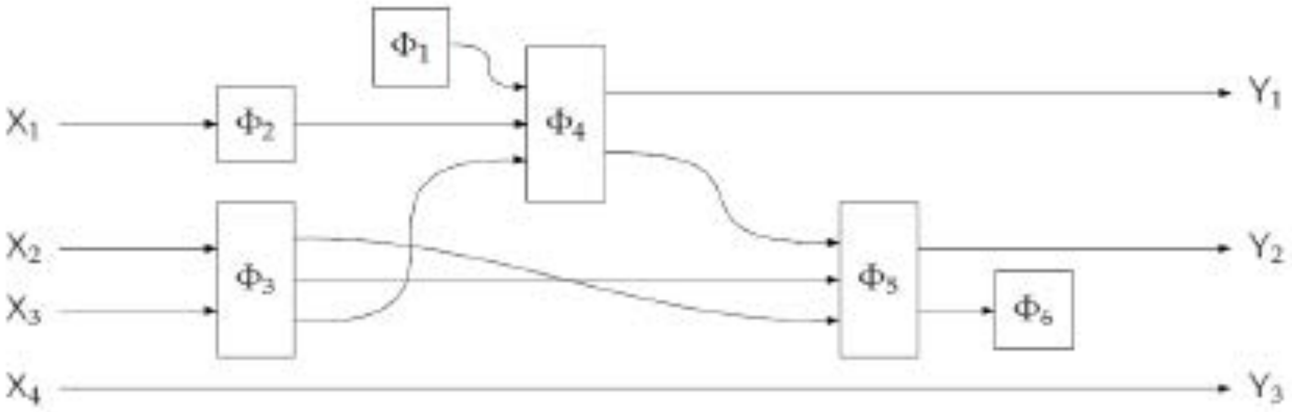
$$|\mathrm{TFD}\rangle = \sum_i e^{-\beta E_i/2}|E_i\rangle_L |E_i\rangle_R$$

$$|\psi(t_L, t_R)\rangle = \sum_i e^{-\beta E_i/2 + iE_i(t_L+t_R)}|E_i\rangle_L |E_i\rangle_R$$

Brown, String 2017

Complexity

- Minimum number of gates required to prepare the desired target state! (one needs to find the optimal circuit)



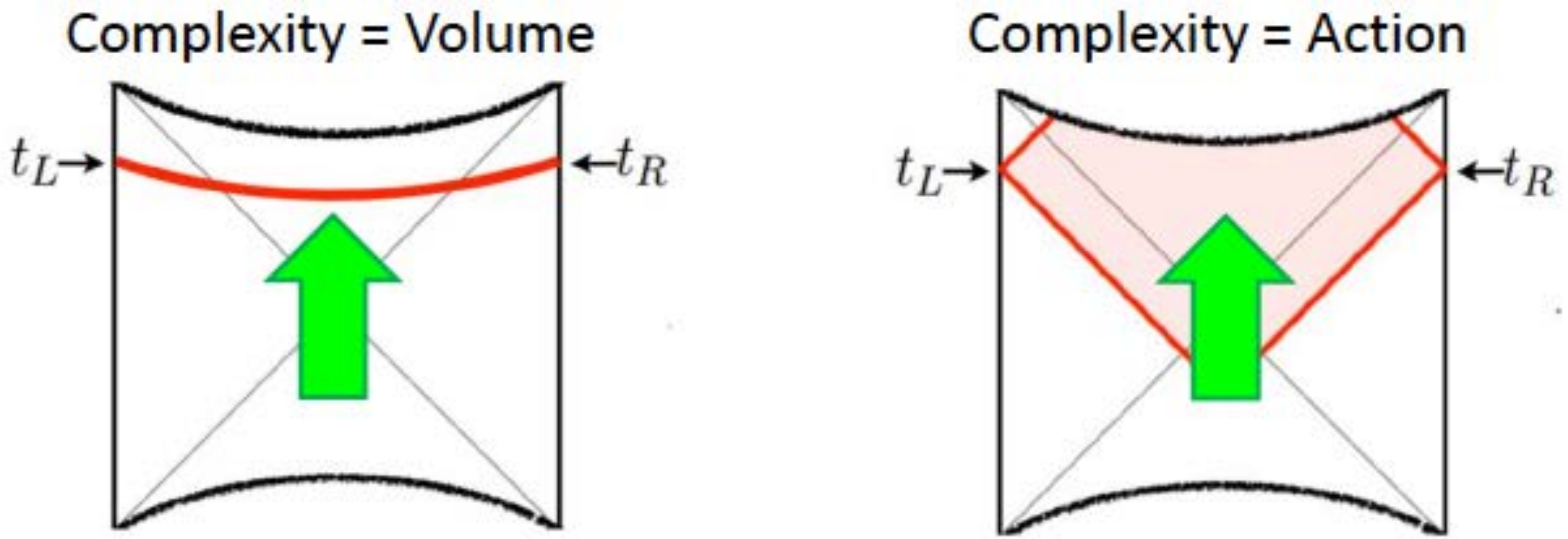
tolerance:

$$\left| \, |\psi\rangle - |\psi\rangle_{\text{Target}} \right|^2 \leq \varepsilon$$

Myers, String 2017

Holographic dictionary for complexity: Complexity=Volume

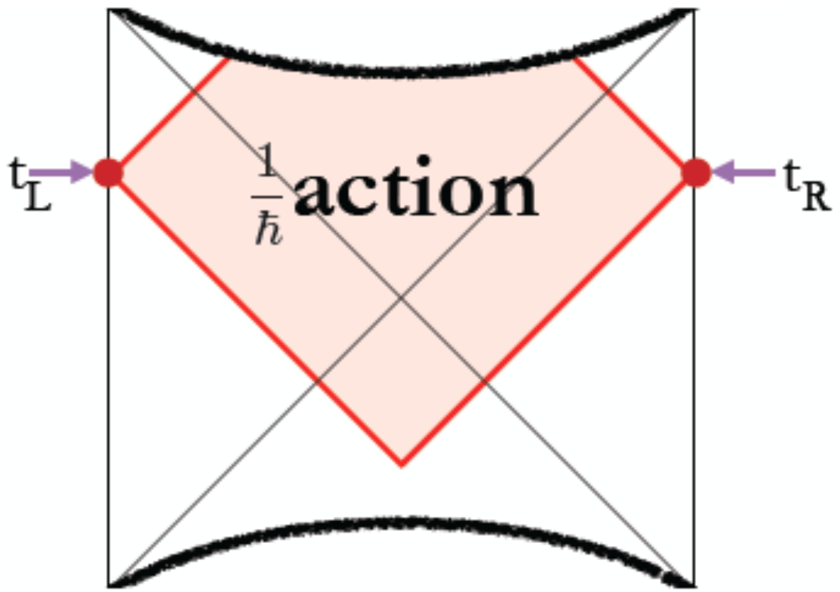
- Evaluate proper volume of extremal codimension-one surface connecting Cauchy surfaces in boundary theory.



Myers, String 2017

Holographic dictionary for complexity: Complexity=Action

- Evaluate gravitational action for Wheeler-DeWitt patch= domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT.



$$\text{Complexity} = \frac{\text{Action}}{\pi\hbar}$$

Entanglement of Purification (EoP)

$$E_P(A:B) = \min_{\rho_{AB}=Tr_{A'B'}|\psi\rangle\langle\psi|} S(\rho_{AA'})$$

$$\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}'_{\mathcal{A}} \otimes \mathcal{H}'_{\mathcal{B}}$$

$$\rho_{AA'} = \mathrm{Tr}_{BB'}[|\psi\rangle\langle\psi|].$$

$$|\psi\rangle \in \mathcal{H}_{AA'} \otimes \mathcal{H}_{BB'}$$

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

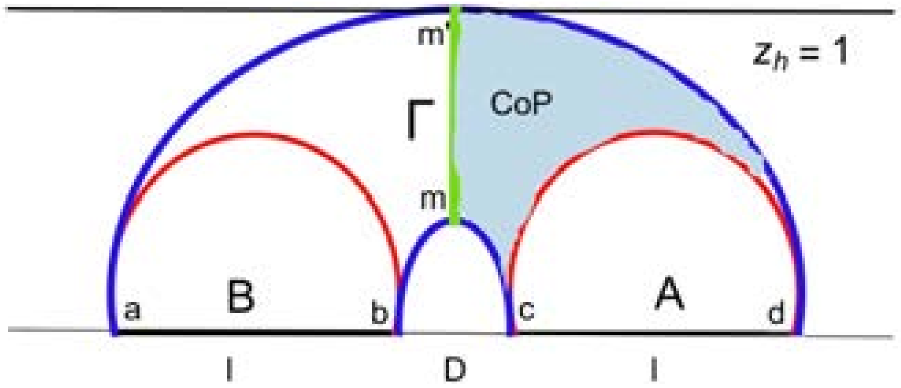
$$\rho_{AB} = \mathrm{Tr}_{A'B'}|\psi\rangle\langle\psi|$$

$$\frac{1}{2} I(A:B) \le E_P(A:B) \le \min\{S(\rho_A), S(\rho_B)\}$$

$$S_A := -tr \rho_A \log \rho_A$$

Entanglement of purification (EoP) for two subregions

$$A := \{l + D/2 > x_1 > D/2, -\infty < x_i < \infty, i = 2, 3, ..., d-1\}$$
$$B := \{-l - D/2 < x_1 < -D/2, -\infty < x_i < \infty, i = 2, 3, ..., d-1\}.$$



$$S_A = S_B = S(l) \qquad S_{AB} = S(2l + D) + S(D)$$

$$I(D, l) = S_A + S_B - S_{AB} = 2S(l) - S(D) - S(2l + D)$$

Background metric: BTZ Black hole

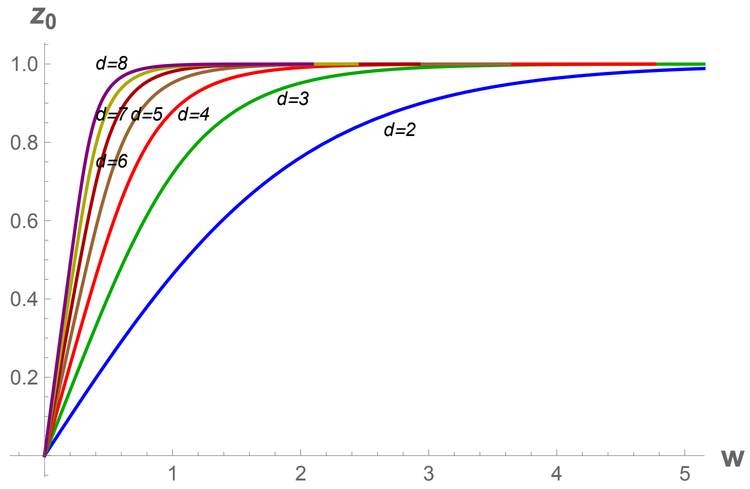
$$ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}^2_{d-1}\right], \qquad f(z) := 1 - z^d/z_h^d$$

$$\sqrt{-g} = \sqrt{x_1'^{\,2} + \frac{1}{f(z)}}\left(\frac{1}{z}\right)^{d-1}, \qquad x_1' = \frac{1}{\sqrt{\left(1-\frac{z^d}{z_h^d}\right)\left(\frac{z_0^{2d-2}}{z^{2d-2}}-1\right)}}.$$

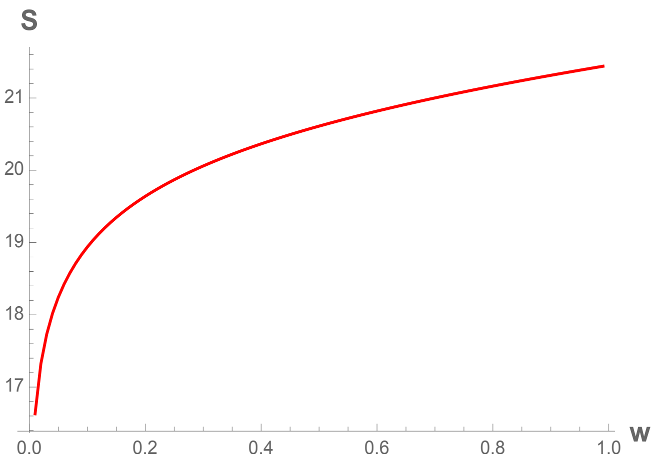
$$w = 2\int_\delta^{z_0} dz \frac{1}{\sqrt{\left(1-\frac{z^d}{z_d^d}\right)\left(\frac{z_0^{2d-2}}{z^{2d-2}}-1\right)}},$$

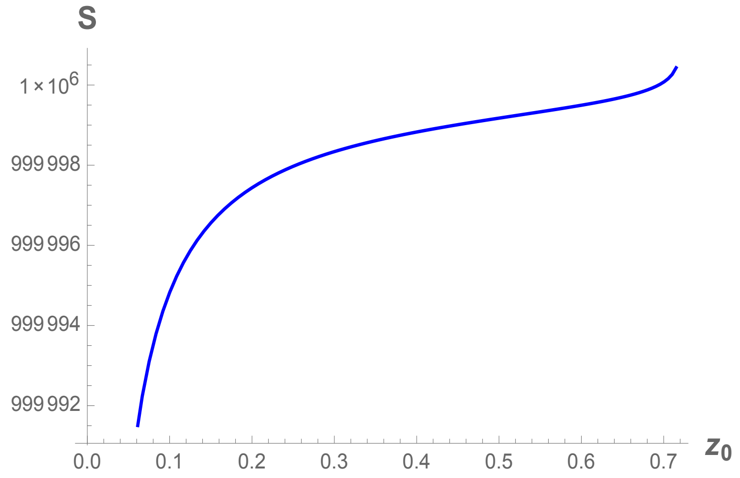
$$S(w) = \frac{2V_{d-2}}{4G_N}\int_\delta^{z_0}\frac{dz}{z^{d-1}}\frac{1}{\sqrt{\left(1-\frac{z^d}{z_h^d}\right)\left(1-\frac{z^{2d-2}}{z_0^{2d-2}}\right)}}.$$

The relationship between turning point and width of the strip



The relationship between entanglement entropy , width of strip and turning point

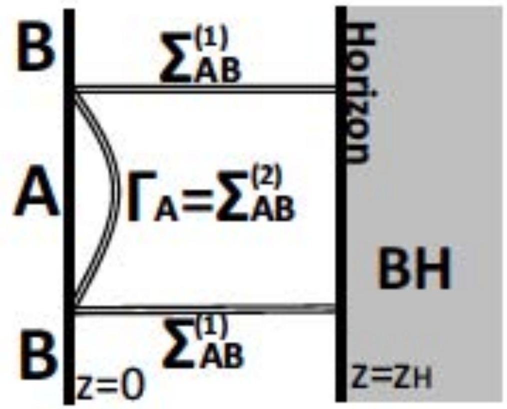




EoP in BTZ Black hole

$$ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + d\vec{x}^2_{d-1}\right], \qquad f(z) := 1 - z^d/z_h^d$$

$$E_W = \frac{c}{3}\min\left[A^{(1)}, A^{(2)}\right]$$



The computation of E_W for BTZ geometry.

Takayanagi-Umemoto 2017

$$A^{(1)} = \log\frac{\beta}{\pi\epsilon} \qquad A^{(2)} = \log\frac{\beta\sinh\left(\frac{\pi l}{\beta}\right)}{\pi\epsilon}$$

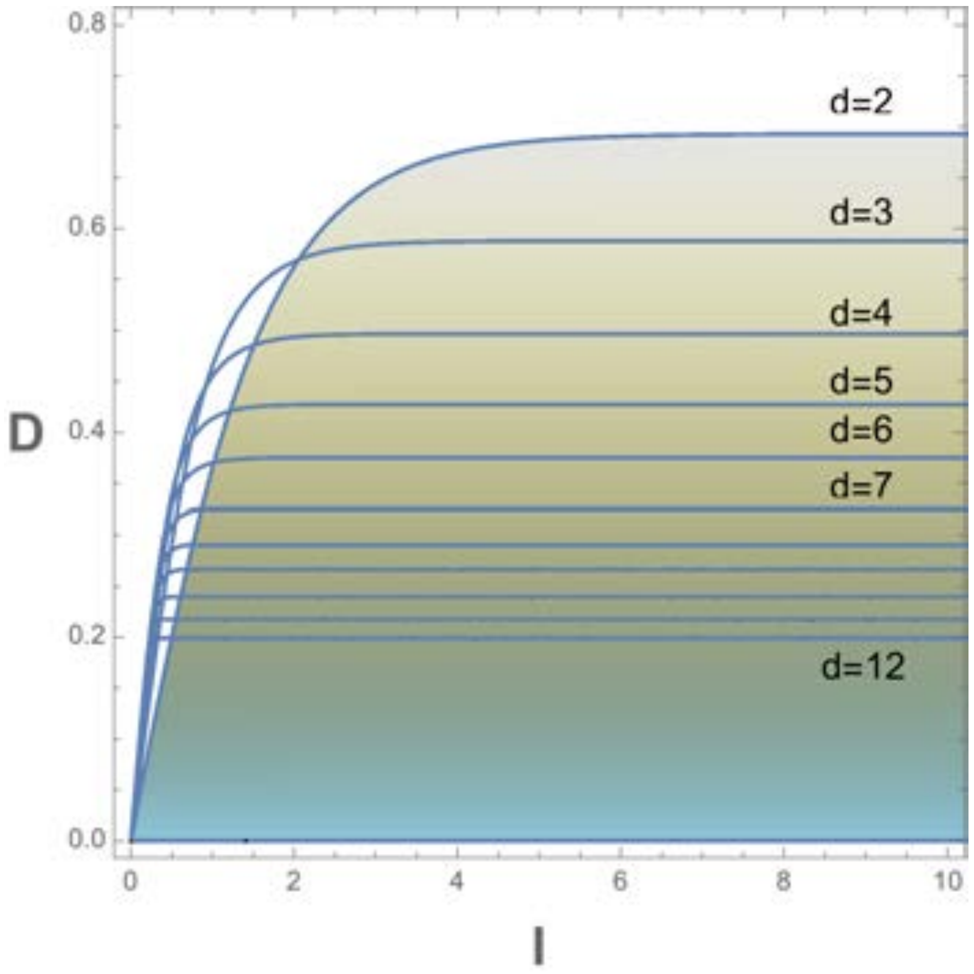
$\Sigma^{(1)}_{AB}$ $\quad l > \beta\log(\sqrt{2}+1)/\pi$

$\Sigma^{(2)}_{AB}$ $\quad l < \beta\log(\sqrt{2}+1)/\pi.$

$$S_A = S_B = S(l)$$

$$S_{AB} = S(2l+D) + S(D) \qquad I(D,l) = S_A + S_B - S_{AB} = 2S(l) - S(D) - S(2l+D)$$

Non-vanishing region of EoP

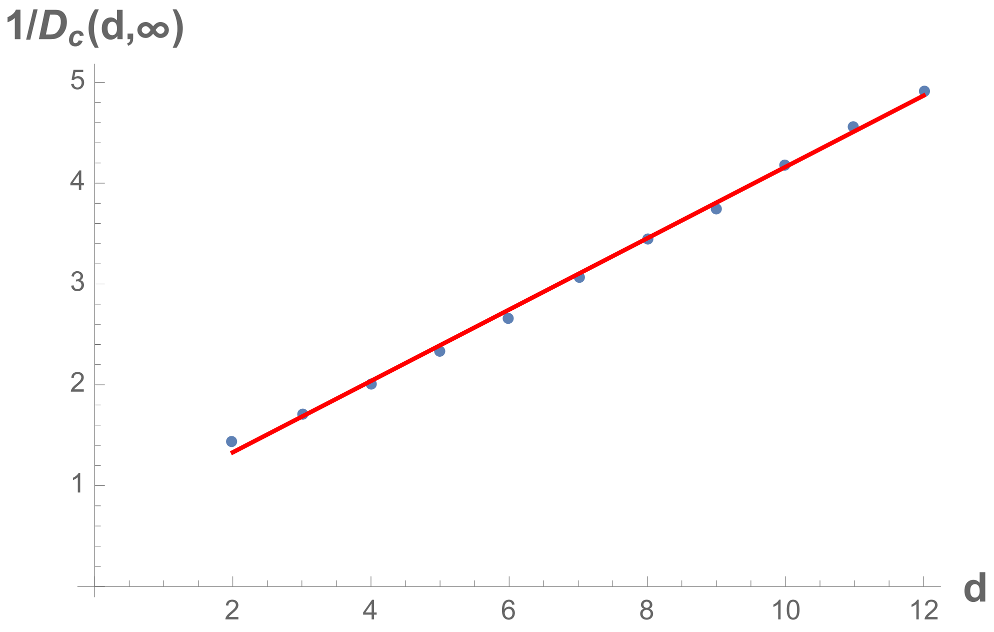


$$I(D,l) = S_A + S_B - S_{AB} = 2S(l) - S(D) - S(2l+D)$$

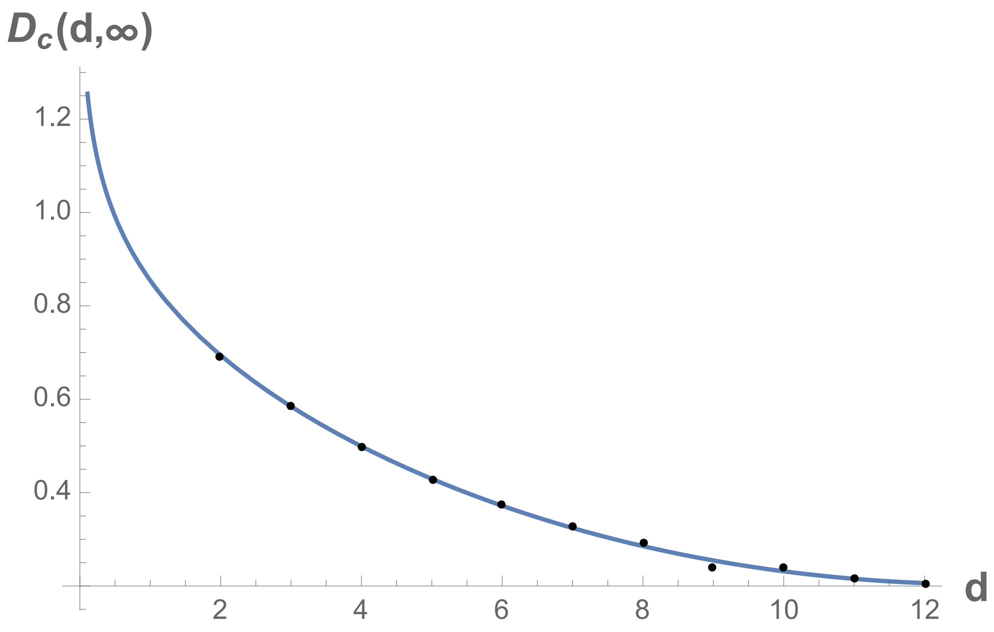
$$\sinh\left(\frac{l}{2}\right)^2 = \sinh\left(\frac{D_c}{2}\right)\sinh\left(\frac{2l+D_c}{2}\right)$$

$$\cosh\frac{D_c(2,l)}{2} = \sqrt{1+2\sqrt{2\cosh l}\cosh\frac{l}{2}+2\cosh l\left[\cosh\frac{3l}{2}-\sqrt{2}(\cosh l)^{3/2}\right]}$$

The relationship between critical D and dimension d



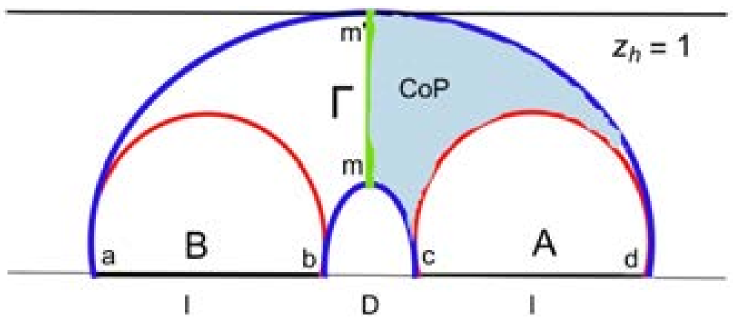
$$D_c(d,\infty)^{-1} \simeq 0.62 + 0.35d.$$



$$D_c(d,\infty) \simeq 0.888 - 0.284\log(d)$$

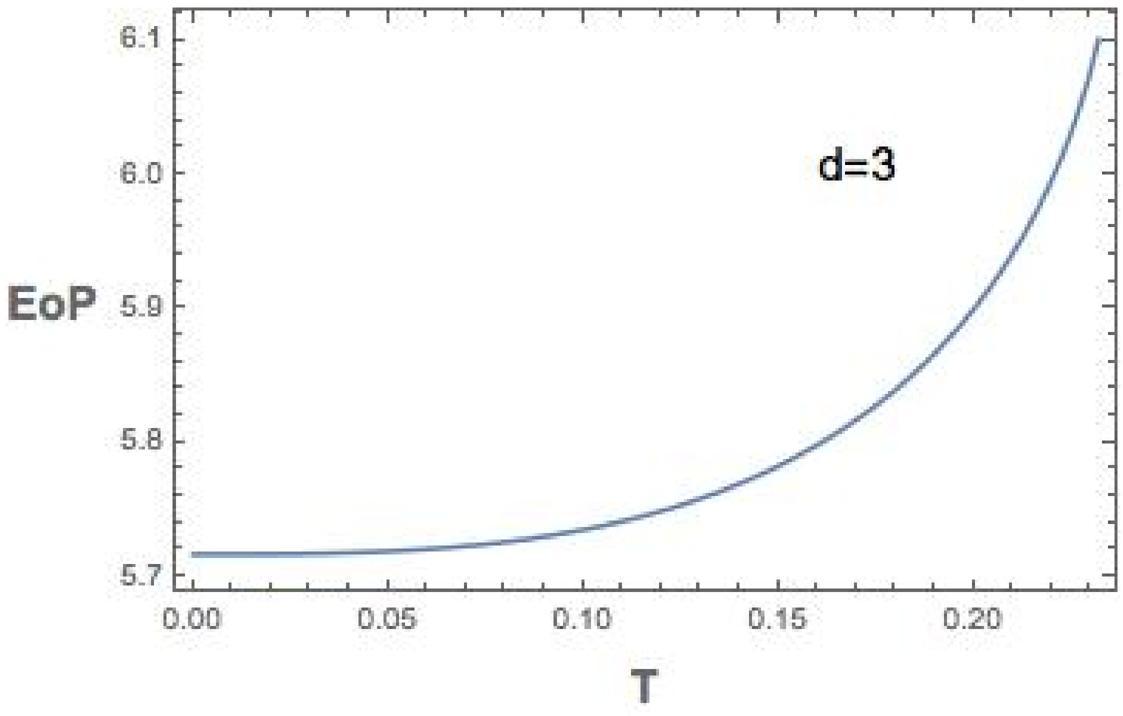
Minimal Wedge cross section and EoP

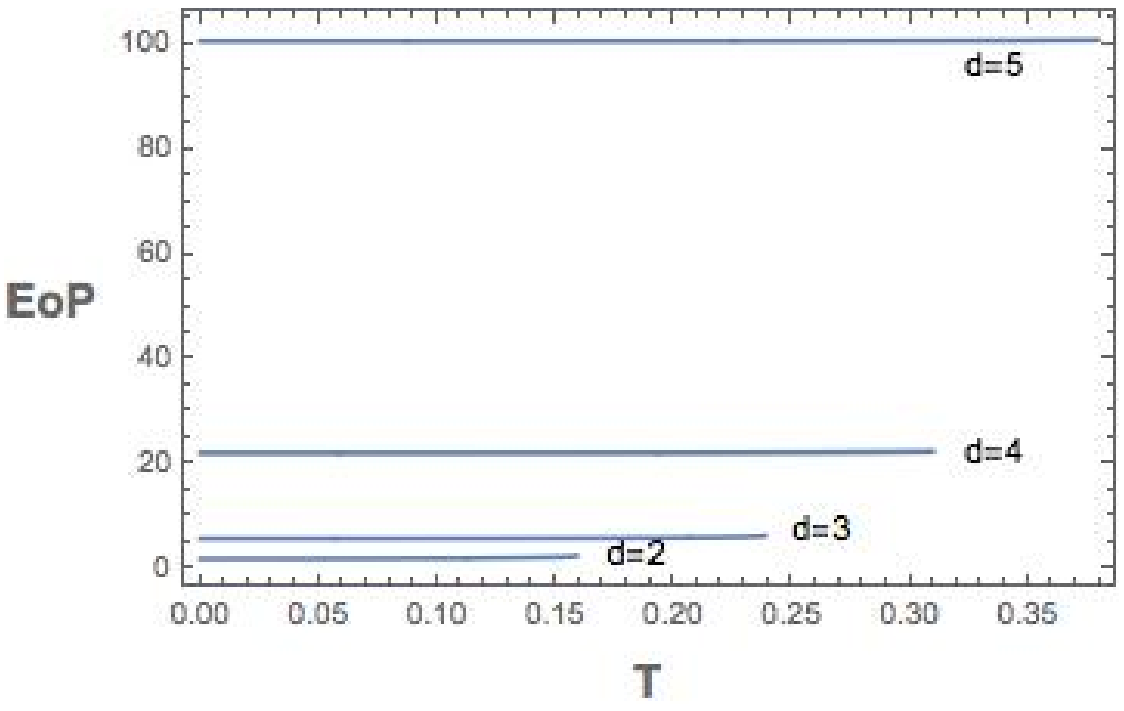
$$\Gamma = \int_{z_D}^{z_{2l+D}} \frac{dz}{z^{d-1}\sqrt{1-\frac{z^d}{z_h^d}}},$$

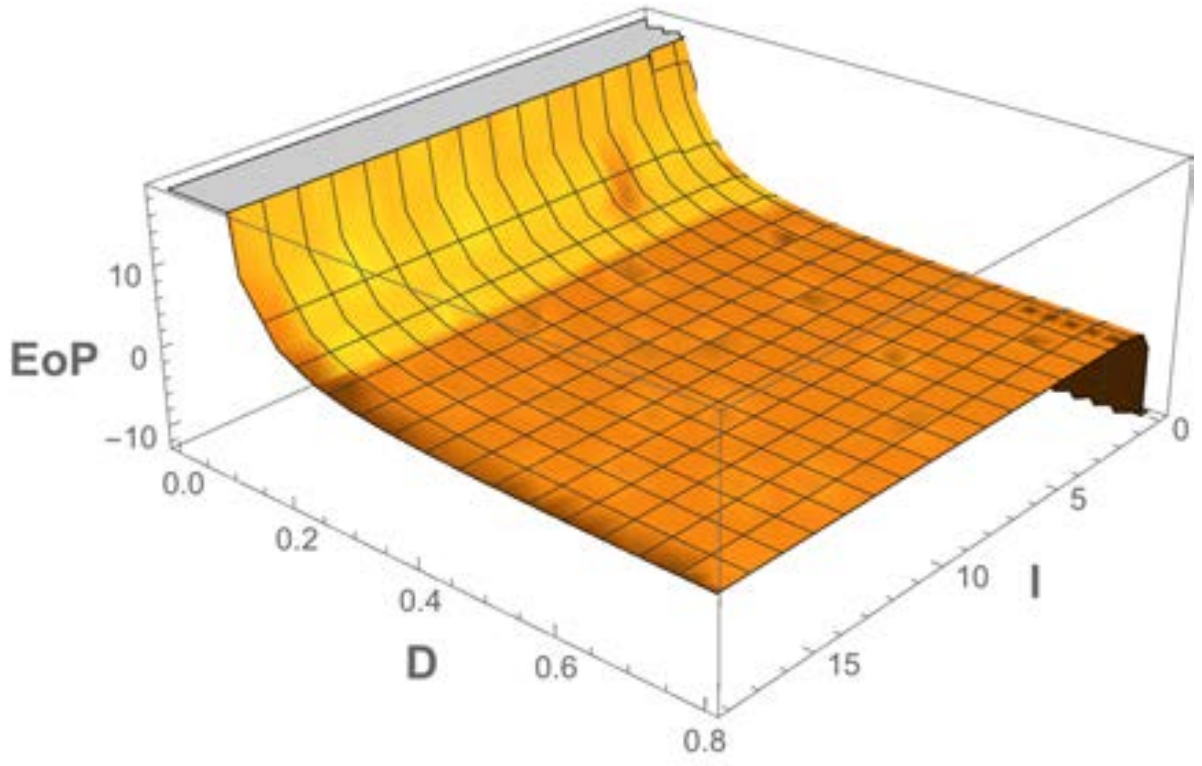


$$\frac{4}{V_{d-2}}E(l,D) = \begin{cases} \ln\frac{\tanh(\frac{D+2l}{2})}{\tanh(\frac{D}{2})} + \ln\frac{1+\sqrt{1-\frac{\tanh^2\left(\frac{D}{2}\right)}{z_h^2}}}{1+\sqrt{1-\frac{\tanh^2\left(\frac{D+2l}{2}\right)}{z_h^2}}}, & d=2, \\ \frac{-4z^{-(d-2)}\sqrt{1-\frac{z^d}{z_h^d}}+(d-4)\frac{z^2}{z_h^d}\,{}_2F_1\left(\frac{1}{2},\frac{2}{d},\frac{d+2}{d},\frac{z^d}{z_h^d}\right)}{4(d-2)}\Bigg|_{z_D}^{z_{2l+D}}, & d>2. \end{cases}$$

The relationship between EoP and Temperature in various dimensions

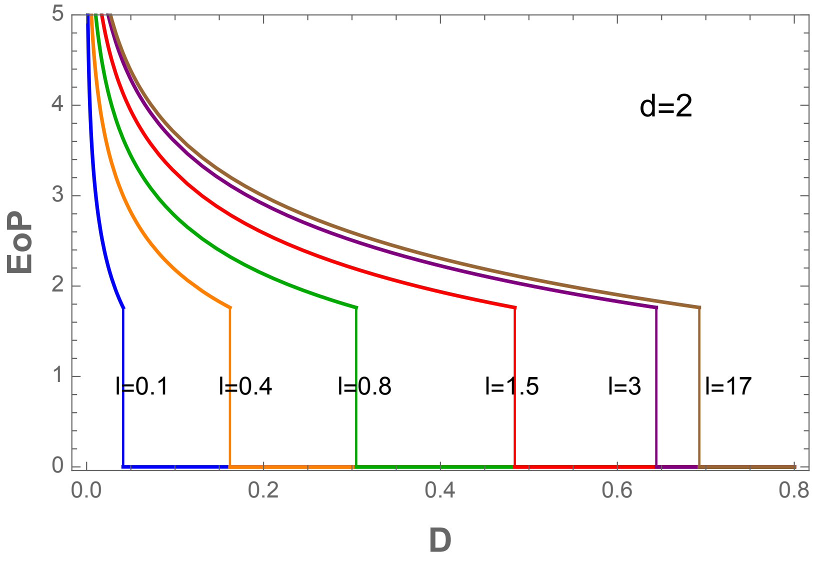


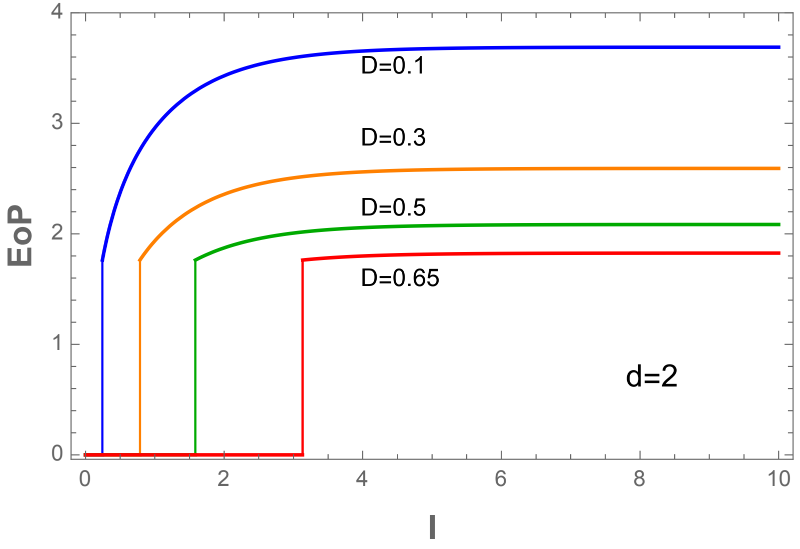


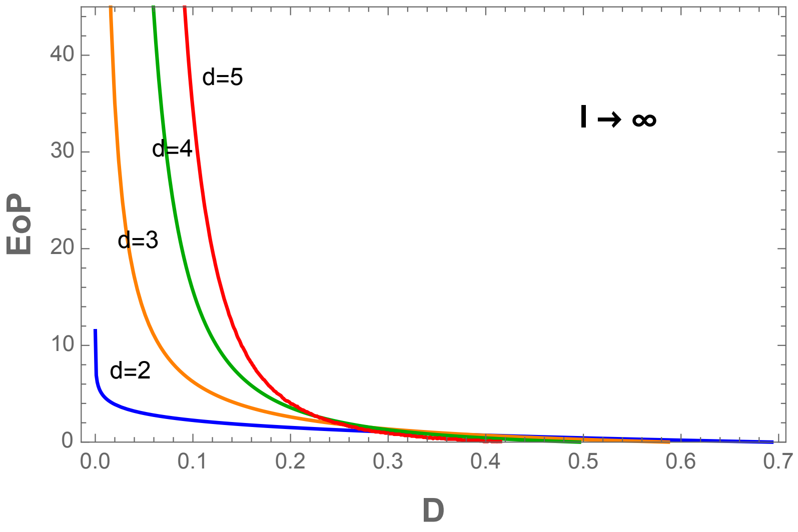


The plot of EoP in three dimensions for different l and D for $d = 4$

The connection between EoP and the distance between strips D and their length l for d=2 Schwarzchild AdS black brane







Monogamy of Mutual Information (MMI)

$I_3(A:BC)$ Is always negative!

$$S(AB)+S(BC)+S(AC)\geq S(A)+S(B)+S(C)+S(ABC)$$

Properties of CoP?!

We choose this! Superadditivity $\mathcal{C}^V(A)+\mathcal{C}^V(B)\leq\mathcal{C}^V(\sigma),$

Supadditivity $\mathcal{C}^A(A)+\mathcal{C}^A(B)\geq\mathcal{C}^A(\sigma)$

Entropy Vector

$$\vec{S} = \{S(A), S(B), S(C), S(AB), S(AC), S(BC), S(ABC)\},$$

$$Q(\vec{S}) = q_A S(A) + q_B S(B) + q_C S(C) +$$
$$q_{AB} S(AB) + q_{AC} S(AC) + q_{BC} S(BC) + q_{ABC} S(ABC),$$

Complexity Vector

$$\vec{\mathcal{C}} = \{\mathcal{C}(A), \mathcal{C}(B), \mathcal{C}(C), \mathcal{C}(AB), \mathcal{C}(AC), \mathcal{C}(BC), \mathcal{C}(ABC)\}$$

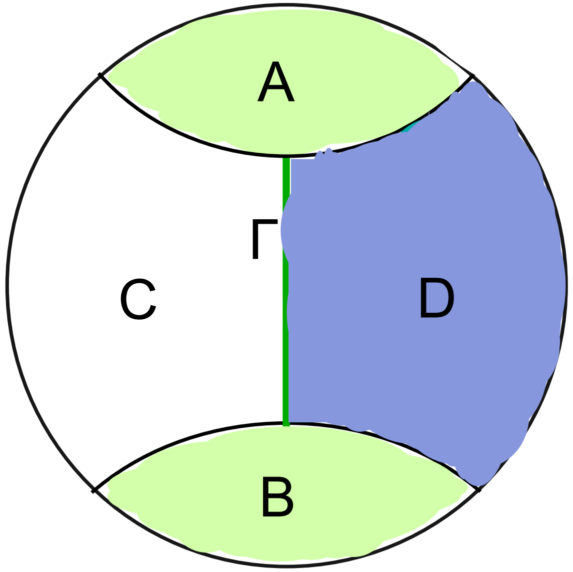
$$Q(\vec{\mathcal{C}}) = q_A \mathcal{C}(A) + q_B \mathcal{C}(B) + q_C \mathcal{C}(C) +$$
$$q_{AB} \mathcal{C}(AB) + q_{AC} \mathcal{C}(AC) + q_{BC} \mathcal{C}(BC) + q_{ABC} \mathcal{C}(ABC),$$

Complexity of purification (CoP) for two subregions

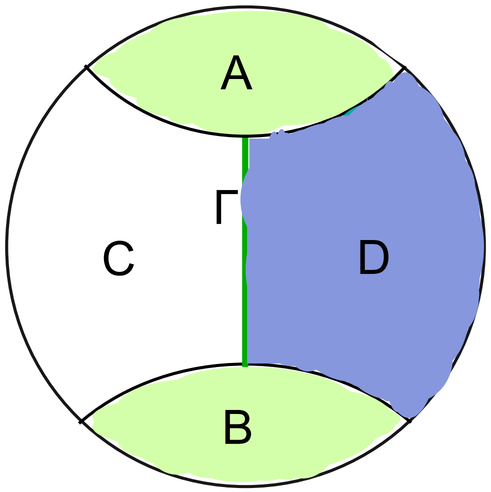
Conditional complexity?

$$\mathcal{C}(A|B) = \mathcal{C}(A) + \mathcal{C}(B) - \mathcal{C}(A \cup B)$$

$$C(A|B) = 2C(l) + C(D) - C(2l + D)$$



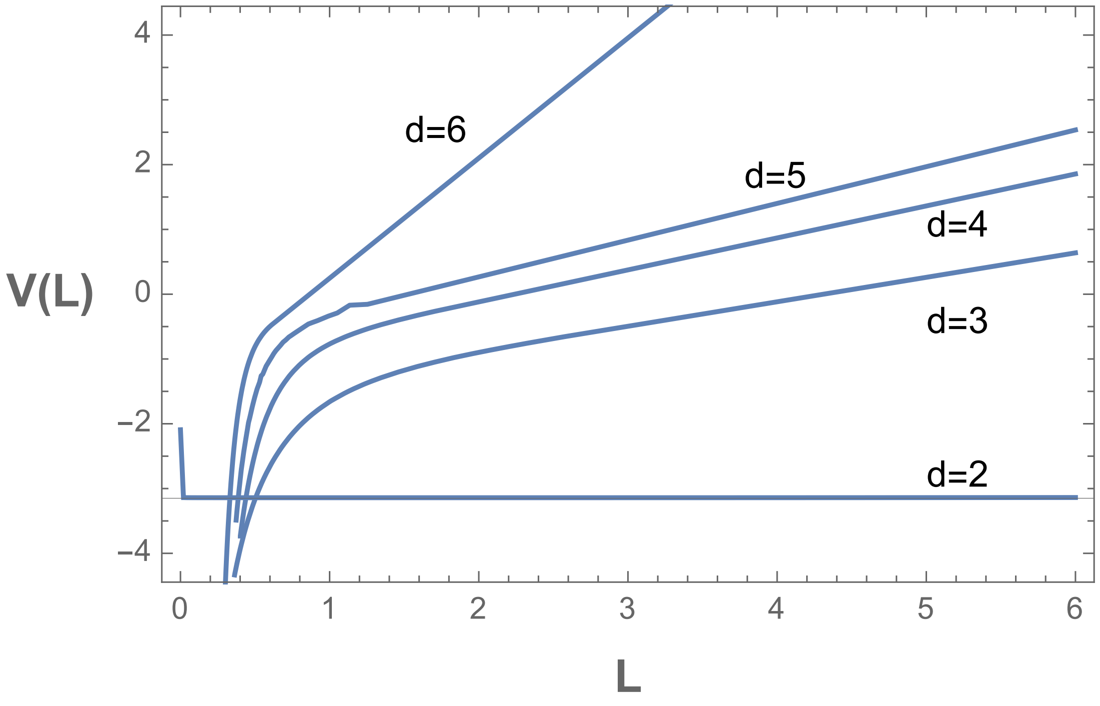
Complexity of purification (CoP) for two subregions

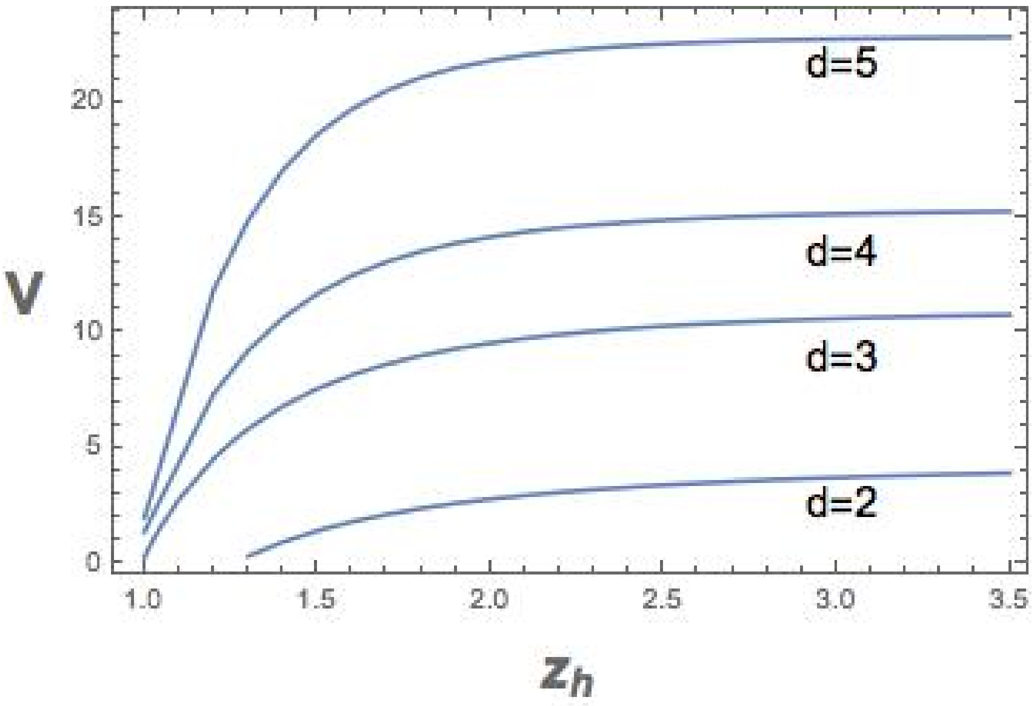


$$CoP(A,B) = \frac{V_D}{8\pi G} = \frac{1}{8\pi G}\left(\frac{V_{ABCD} - V_A - V_B}{2}\right)$$

$$CoP \sim \frac{1}{2}\big(C(2l+D) - 2C(l) - C(D)\big) = -\frac{1}{2}C(A|B).$$

The relationship between the Volume and the length of one strip



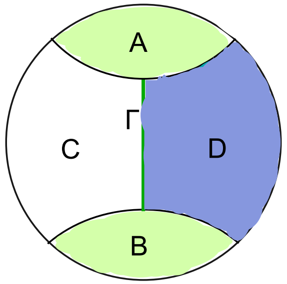


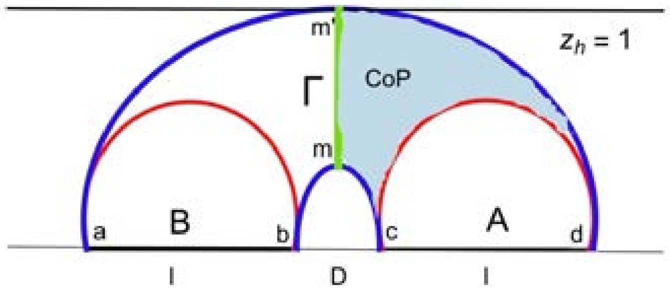
The equation for CoP

$$V_D = 2L^{d-2}\Bigg(\int_\delta^{z_{2l+D}} \frac{dz}{z^d\sqrt{1-z^d}}\int_z^{z_{2l+D}} \frac{dZ}{\sqrt{(1-Z^d)(\frac{z_{2l+D}^{2d-2}}{Z^{2d-2}}-1)}}$$

$$-\int_\delta^{z_D} \frac{dz}{z^d\sqrt{1-z^d}}\int_z^{z_D} \frac{dZ}{\sqrt{(1-Z^d)(\frac{z_D^{2d-2}}{Z^{2d-2}}-1)}}$$

$$-2\int_\delta^{z_l} \frac{dz}{z^d\sqrt{1-z^d}}\int_z^{z_l} \frac{dZ}{\sqrt{(1-Z^d)(\frac{z_l^{2d-2}}{Z^{2d-2}}-1)}}\Bigg)$$



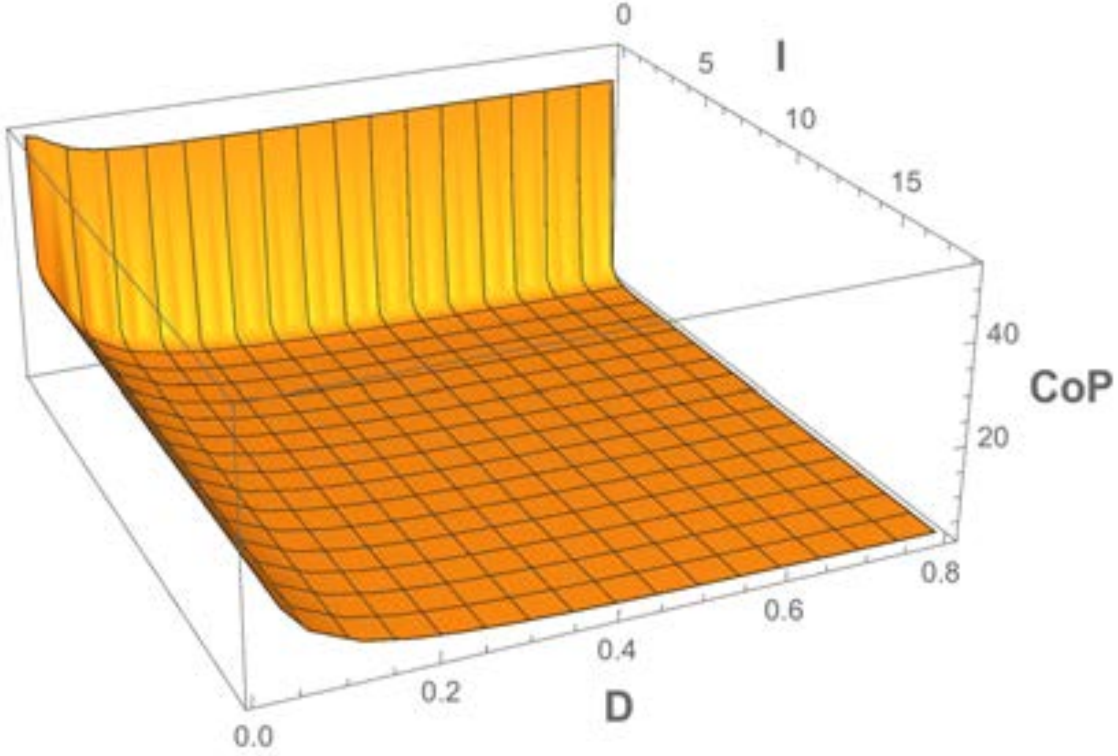


$$V_D = \left(-\pi - \frac{1}{\delta}\operatorname{arctanh}\left(\frac{1}{z_{2l+D}}\right)\right) - \left(-\pi - \frac{1}{\delta}\operatorname{arctanh}\left(\frac{1}{z_D}\right)\right) - 2\left(-\pi - \frac{1}{\delta}\operatorname{arctanh}\left(\frac{1}{z_l}\right)\right)$$

$$= 2\pi + \frac{1}{\delta}\left[2\operatorname{arctanh}\left(\coth\left(\frac{l}{2}\right)\right) + \operatorname{arctanh}\left(\coth\left(\frac{D}{2}\right)\right) - \operatorname{arctanh}\left(\coth\left(\frac{2l+D}{2}\right)\right)\right]$$

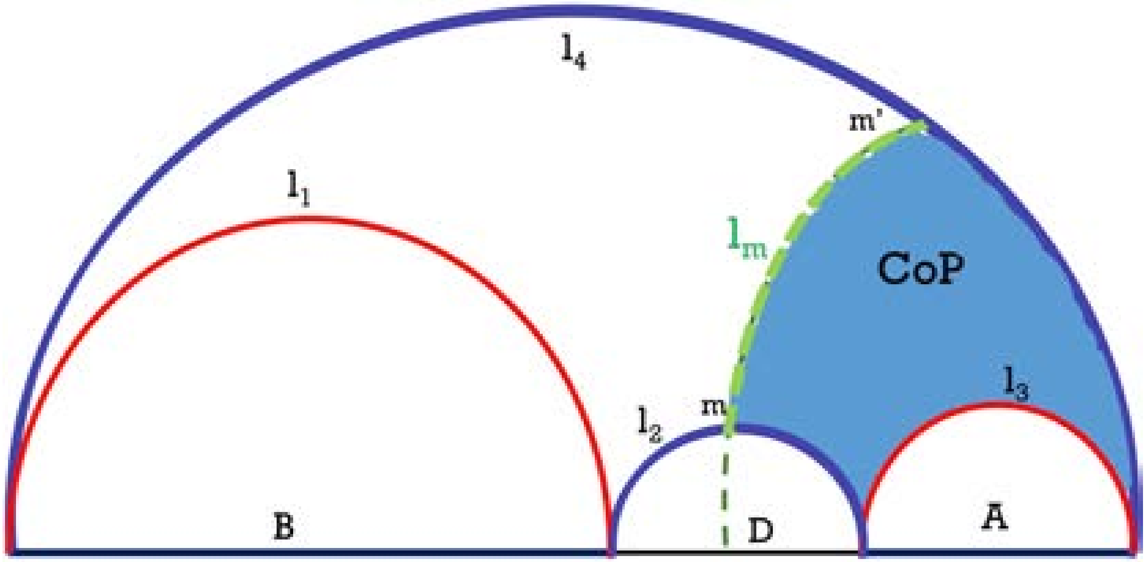
$$= 2\pi - \frac{i\pi}{\delta}$$

d=2



The relationship between complexity of purification and D, l for $d = 3$

CoP for non-symmetrical systems



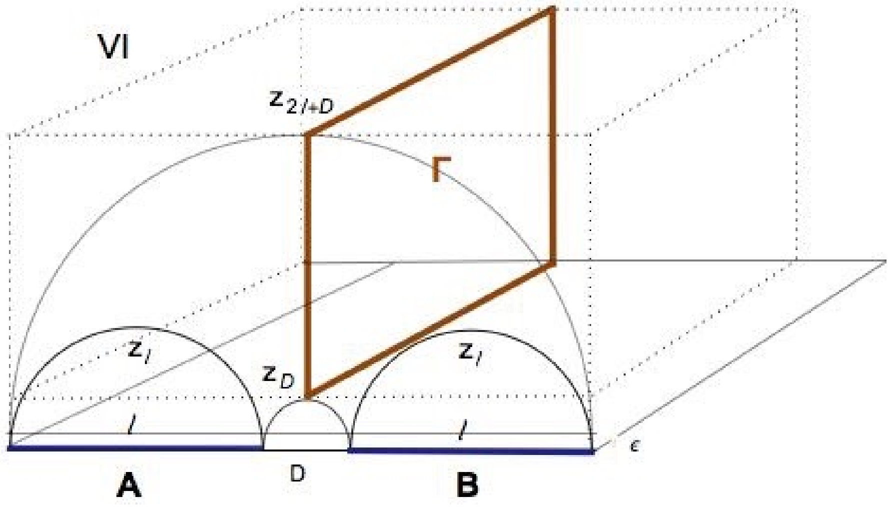
arXiv:1902.02243 P. Liu, Y. Ling, C. Niu, and J.-P. Wu

The new measure: The Interval Volume (VI)

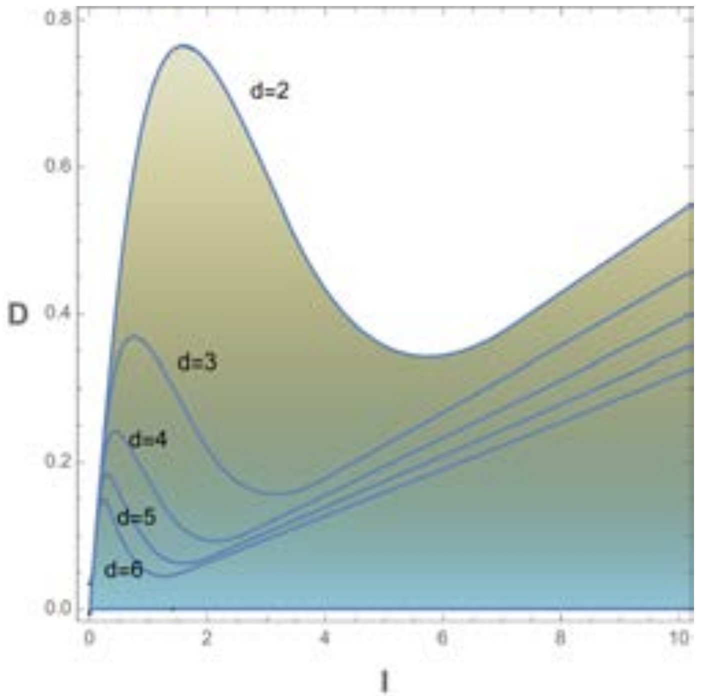
$$VI = \frac{1}{2}\left(\int_{\epsilon}^{z_{2l+D}} \frac{dz}{z^d\sqrt{f(z)}} - \int_{\epsilon}^{z_D} \frac{dz}{z^d\sqrt{f(z)}} - 2\int_{\epsilon}^{z_l} \frac{dz}{z^d\sqrt{f(z)}}\right)$$

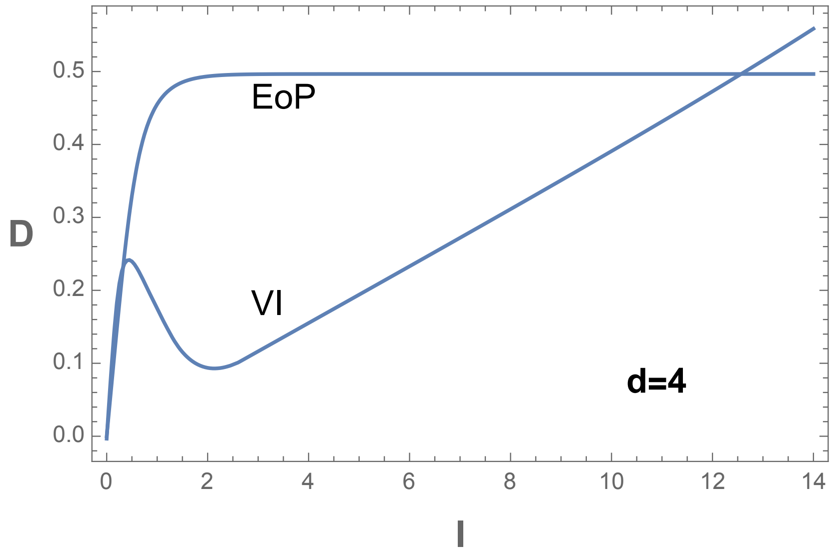
$$G(z) \equiv \int_0^z \frac{dz}{z^d\sqrt{f(z)}} = \frac{-2z^{1-d}\sqrt{1-z^d} + z(d-2)\,{}_2F_1\left(\frac{1}{2}, \frac{1}{d}, \frac{d+1}{d}, z^d\right)}{2(d-1)}$$

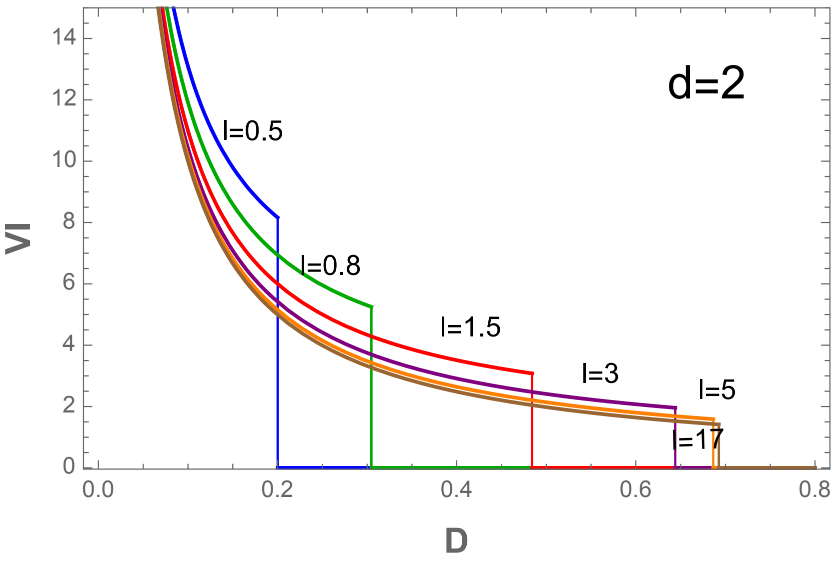
$$VI = \frac{1}{2}\left(G(z_{2l+D}) - G(z_D)\right) - G(z_l) + G(\epsilon)$$

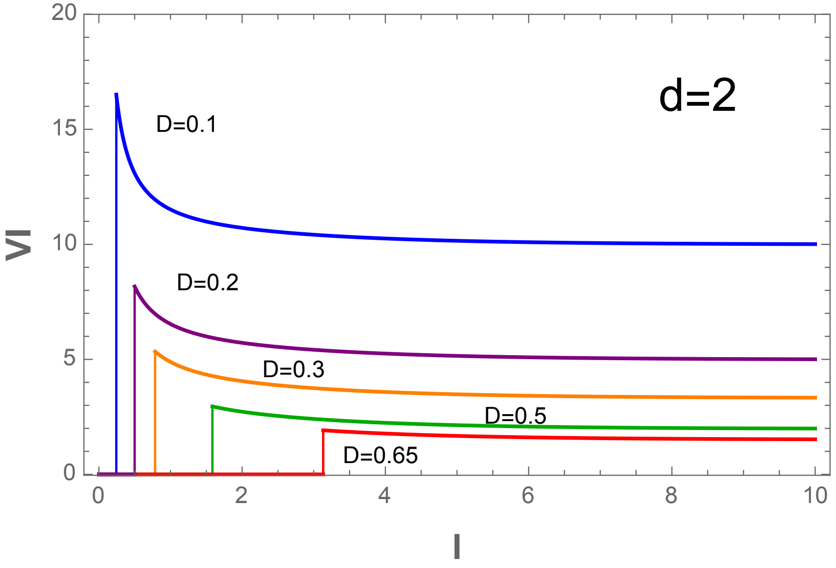


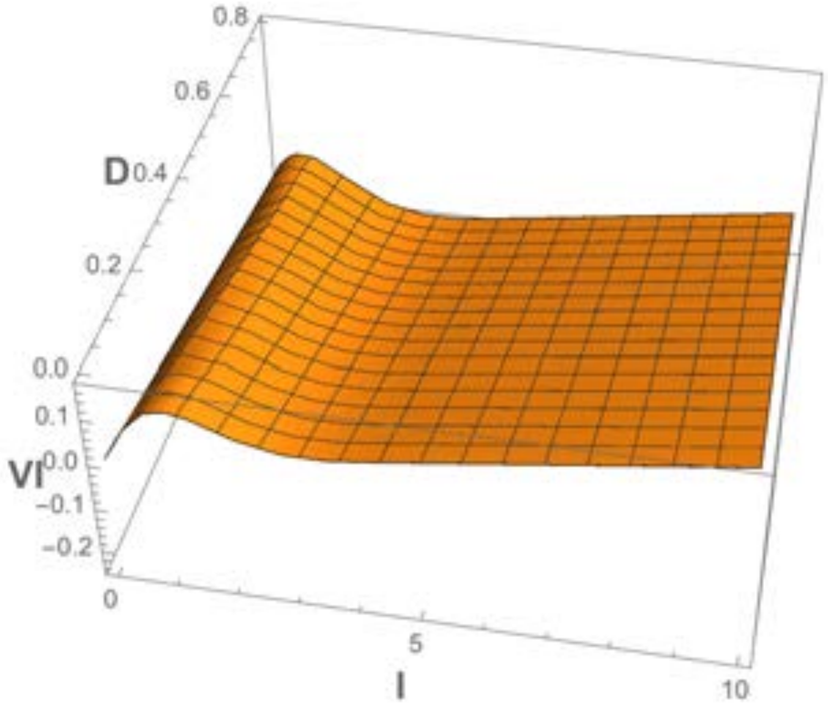
$$\frac{4}{V_{d-1}} C_E(l,D) = \begin{cases} \frac{1}{2}\Big(\mathrm{csch}(\frac{D}{2}) + 2\mathrm{csch}(\frac{l}{2}) - \mathrm{csch}(\frac{2l+D}{2})\Big), & d=2, \\ \\ \frac{1}{2}G(z_{2l+D}) - \frac{1}{2}G(z_D) - G(z_l), & d>2. \end{cases}$$


D
I
d=2
d=3
d=4
d=5
d=6


D
I
EoP
VI
d=4


VI
D
d=2
l=0.5
l=0.8
l=1.5
l=3
l=5
l=17


VI
I
d=2
D=0.1
D=0.2
D=0.3
D=0.5
D=0.65



$$\text{Positivity:} \qquad C_A^P > 0,$$
$$\text{Monotonicity:} \qquad C_{A+\delta A}^P > C_A^P,$$
$$\text{Weak Superadditivity:} \qquad C_A^P + C_{\delta A}^P < 2C_{A+\delta A}^P$$

Purification of BTZ black hole solution in massive gravity theory

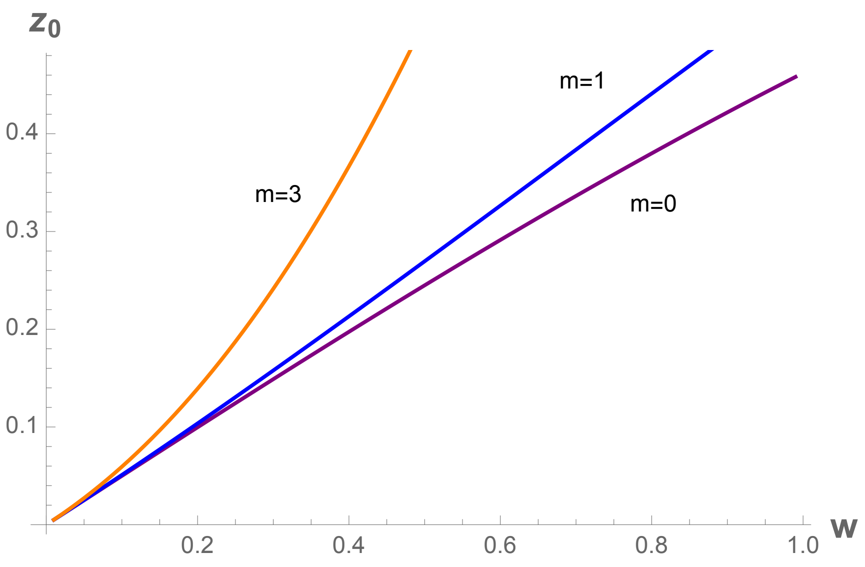
$$ds^2 = \frac{1}{z^2}[-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2] \quad \text{with} \quad f(z) = 1 - z^2 + m^2 c c_1 z$$

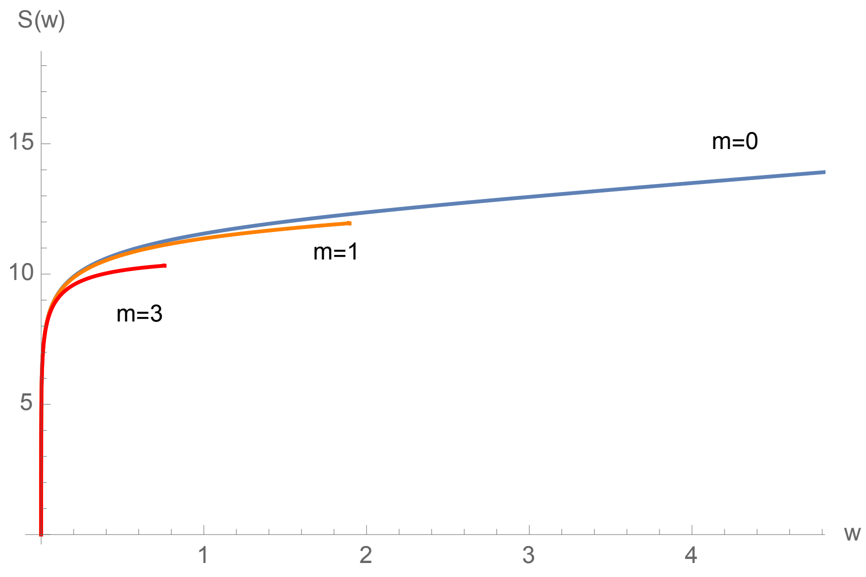
$$\mathcal{I} = -\frac{1}{16\pi}\int d^3x\sqrt{-g}\left[\mathcal{R} + 2 + m^2\sum_i^4 c_i\mathcal{U}_i(g,h)\right]$$

Again, finding the relationship between turning point, width of the strip and entropy gives:

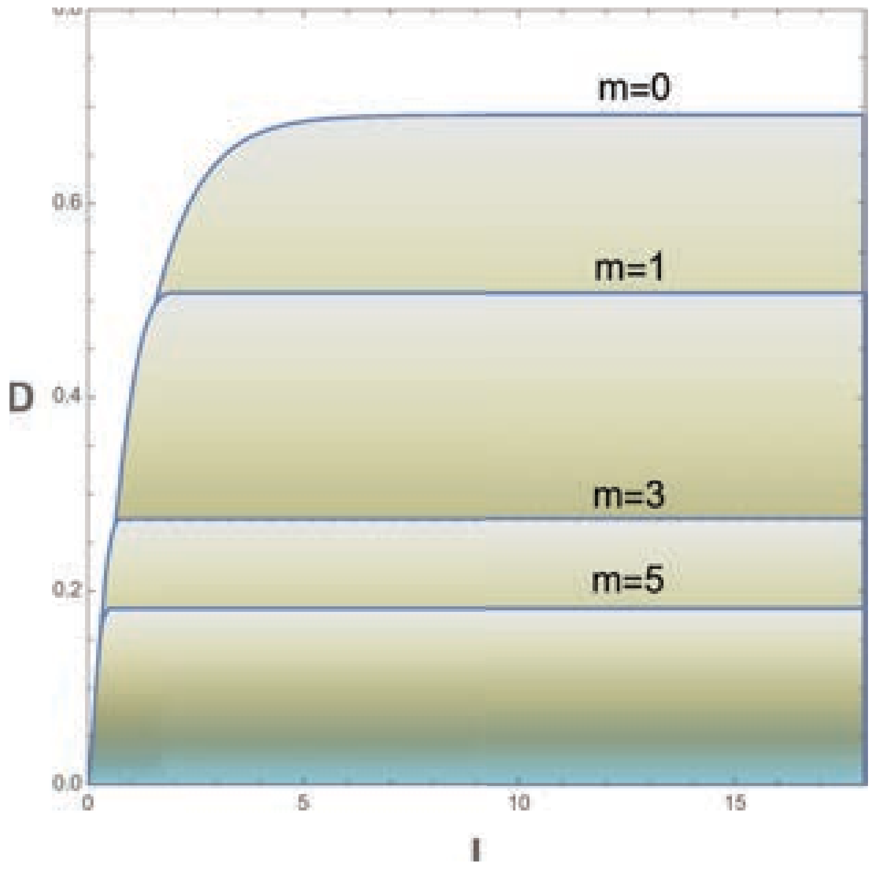
$$w = 2\int_{\delta}^{z_0} dz \frac{1}{\sqrt{f\left(\frac{z_0^2}{z^2}-1\right)}},$$

$$S(\omega) = \frac{1}{2}\int_{\delta}^{z_0} \frac{dz}{z} \frac{1}{\sqrt{f(z)\left(1-\frac{z^2}{z_0^2}\right)}}.$$



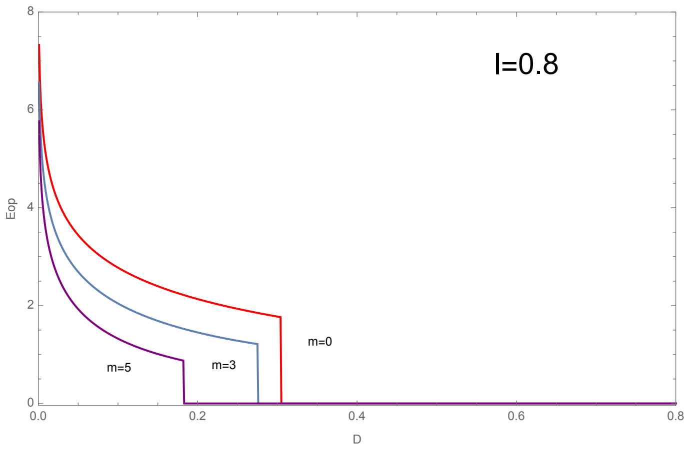


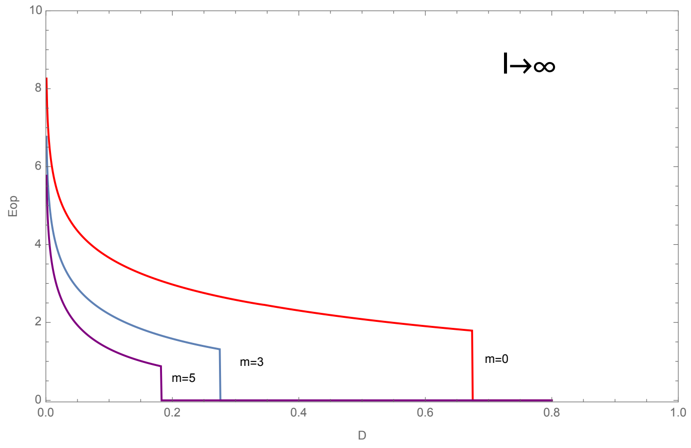
EoP in massive BTZ

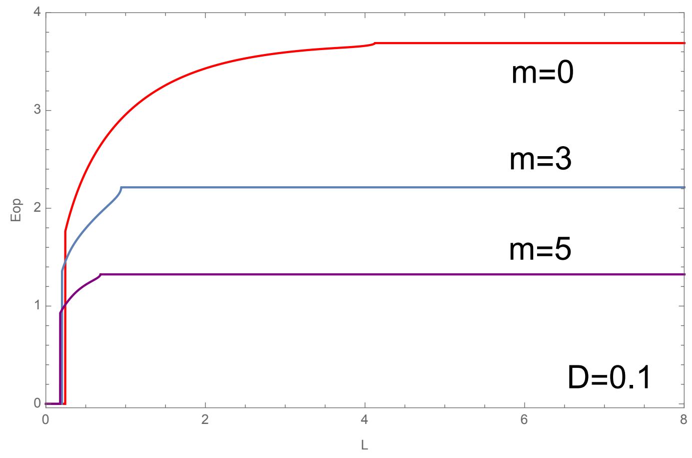


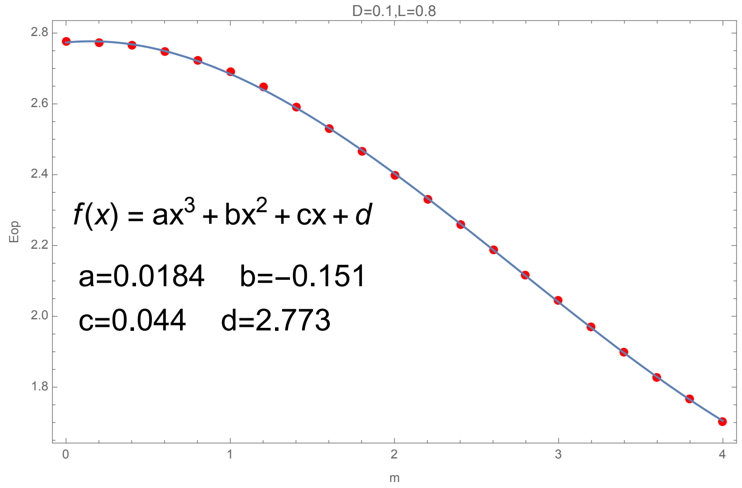
$$\Gamma = \int_{z_D}^{z_{2l+D}} \frac{dz}{z\sqrt{1 - z^2 + m^2 c c_1 z}},$$

$$2E(l,D) = \frac{\log z}{\log(2 + m^2 z + 2\sqrt{1 + (m^2 - z)z})}\Bigg|_{z_D}^{z_{2l+D}}$$





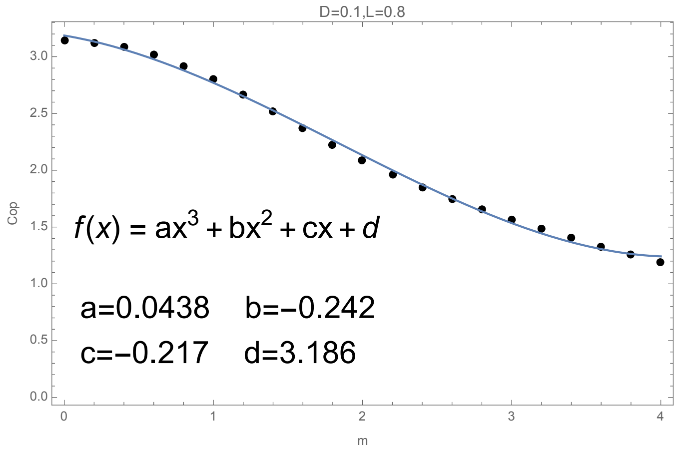




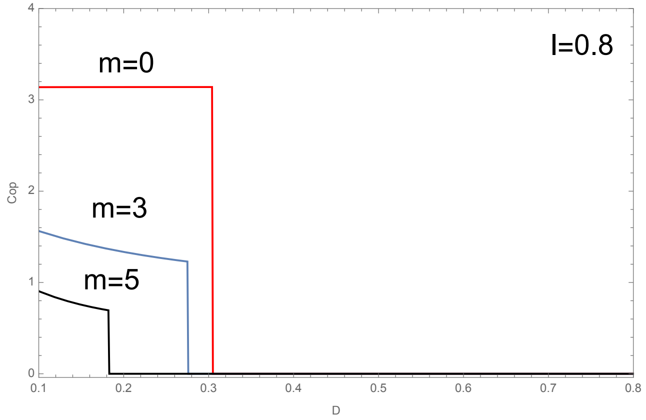
EoP as function of m with fixed $D = 0.1$ and $l = 0.8$

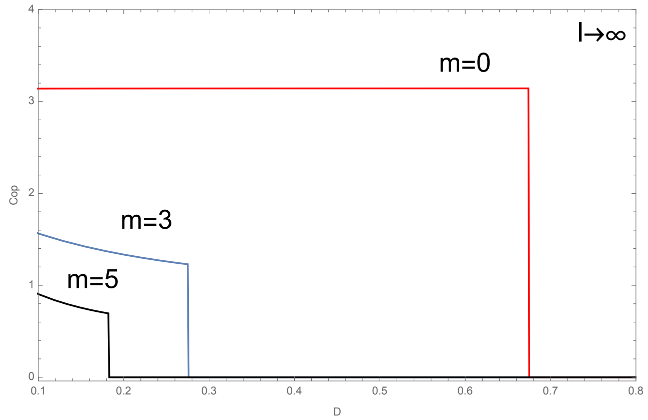
CoP in massive BTZ

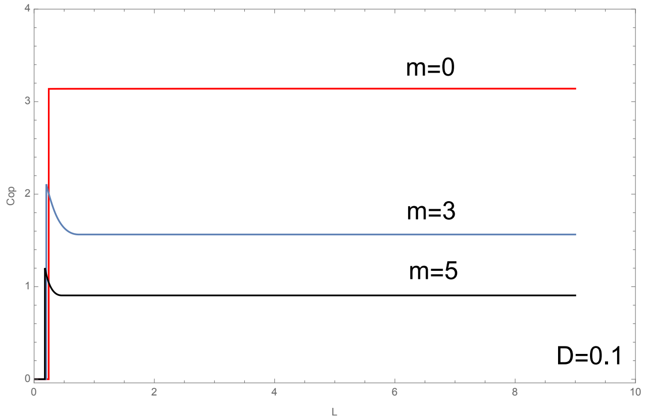
$$
\begin{aligned}
CoP = & \int_{\delta}^{z_{2l+D}} \frac{dz}{z^2\sqrt{1-z^2+m^2cc_1z}} \int_{z}^{z_{2l+D}} \frac{dZ}{\sqrt{(1-z^2+m^2cc_1z)(z_{2l+D}^2/Z^2-1)}} \\
& - \int_{\delta}^{z_D} \frac{dz}{z^2\sqrt{1-z^2+m^2cc_1z}} \int_{z}^{z_D} \frac{dZ}{\sqrt{(1-z^2+m^2cc_1z)(z_D^2/Z^2-1)}} \\
& - 2\int_{\delta}^{z_l} \frac{dz}{z^2\sqrt{1-z^2+m^2cc_1z}} \int_{z}^{z_l} \frac{dZ}{\sqrt{(1-z^2+m^2cc_1z)(z_l^2/Z^2-1}},
\end{aligned}
$$



CoP as function of m with fixed $D = 0.1$ and $l = 0.8$



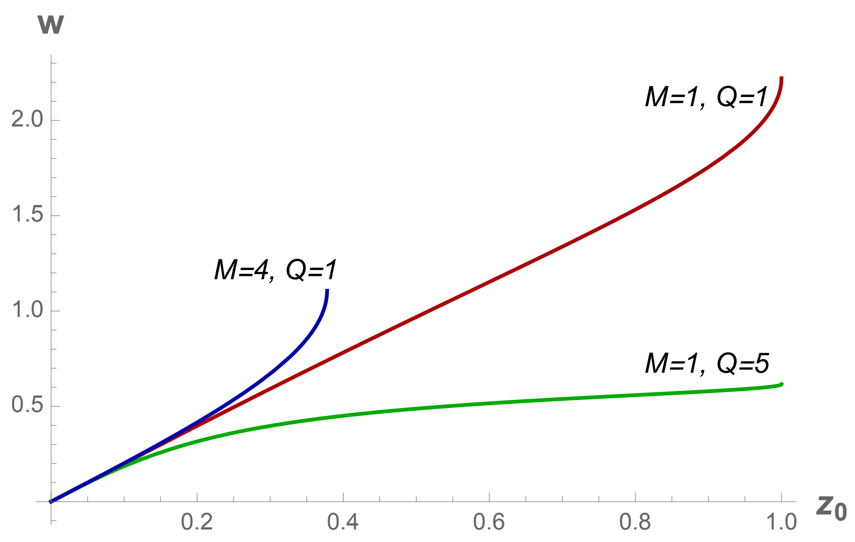


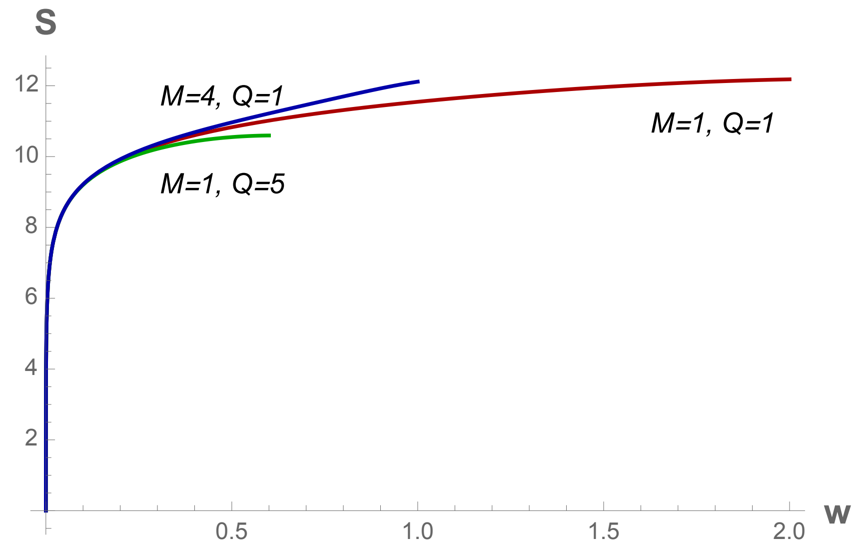


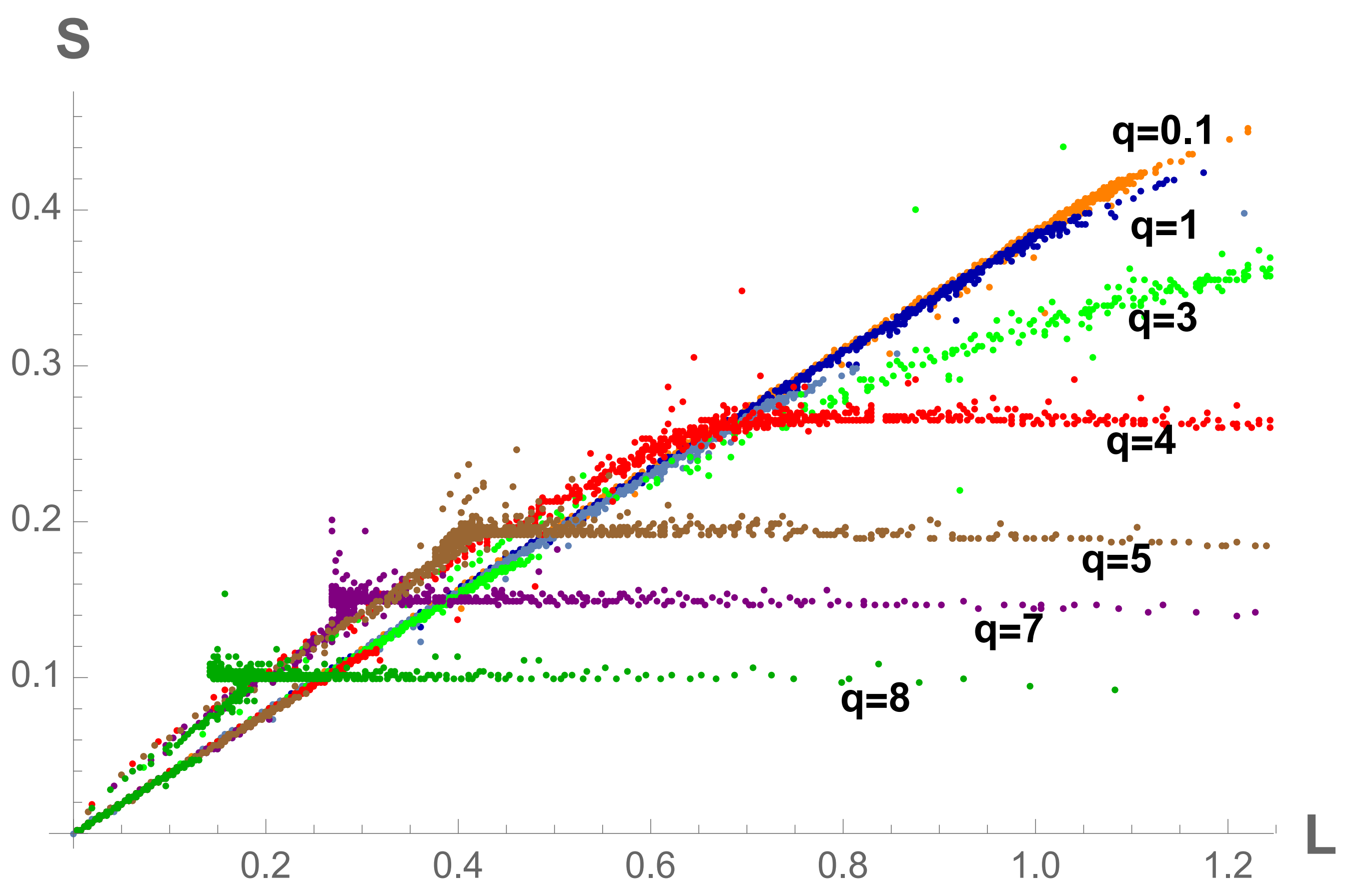
Purification of charged BTZ black hole

$$ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2\right]$$
$$f(z) = 1 - z^2 + \frac{Q^2}{2} z^2\, ln(z)$$
$$A = Q\, ln(z)\, dt$$

$$w = 2\int_\delta^{z_0} \frac{dz}{\sqrt{f}} \frac{1}{\sqrt{\frac{z_0^{2d-2}}{z^{2d-2}} - 1}},$$
$$S = \frac{2V_{d-2}}{4G_N}\int_\delta^{z_0} \frac{dz}{z^{d-1}} \frac{1}{\sqrt{f}} \frac{1}{\sqrt{1 - \frac{z^{2d-2}}{z_0^{2d-2}}}}.$$

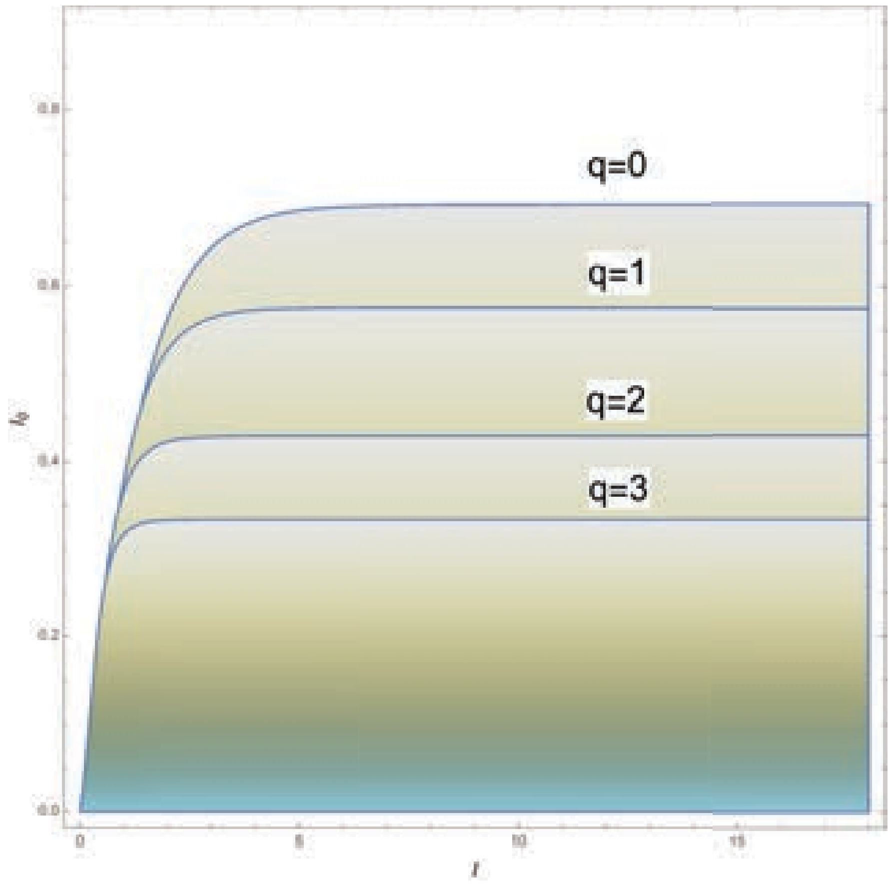






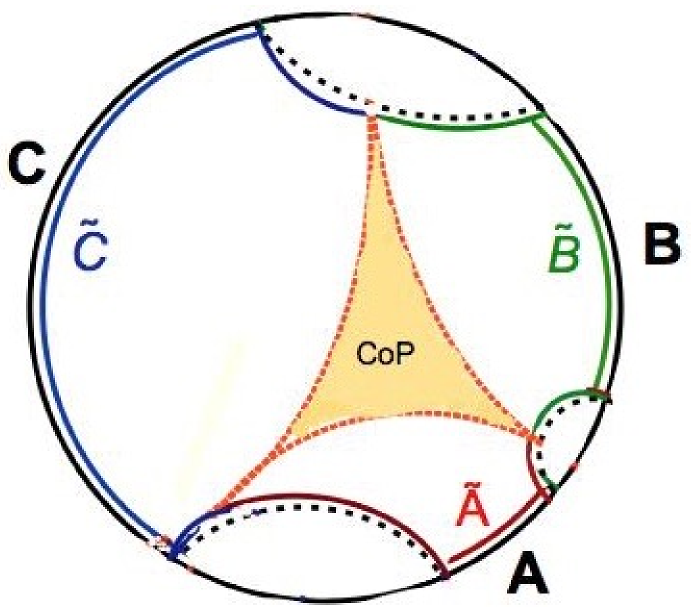
The relationship between $S(w)$ and w for charged BTZ black hole.

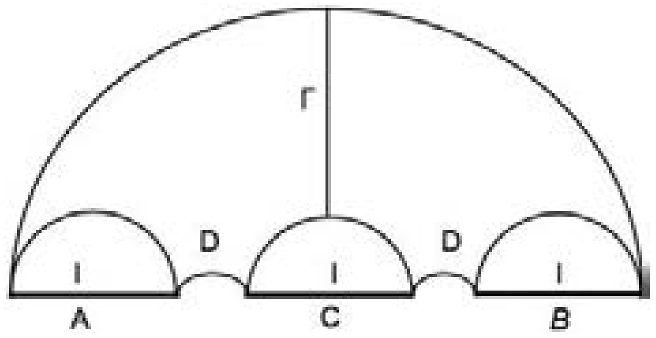
$$\Gamma = \int_{z_D}^{z_{2l+D}} \frac{dz}{z\sqrt{1 - z^2 + 2q^2 z^2 \ln \frac{1}{z}}}.$$



Purification of multipartite systems

$$\partial M_{ABC} = A \cup B \cup C \cup \Sigma_{ABC}^{min}.$$





$$CoP_{A,B}\left((n+1)l + nD\right) = 2n\pi - \frac{1}{\delta}(ni\pi),$$

Operational and bit thread interpretations

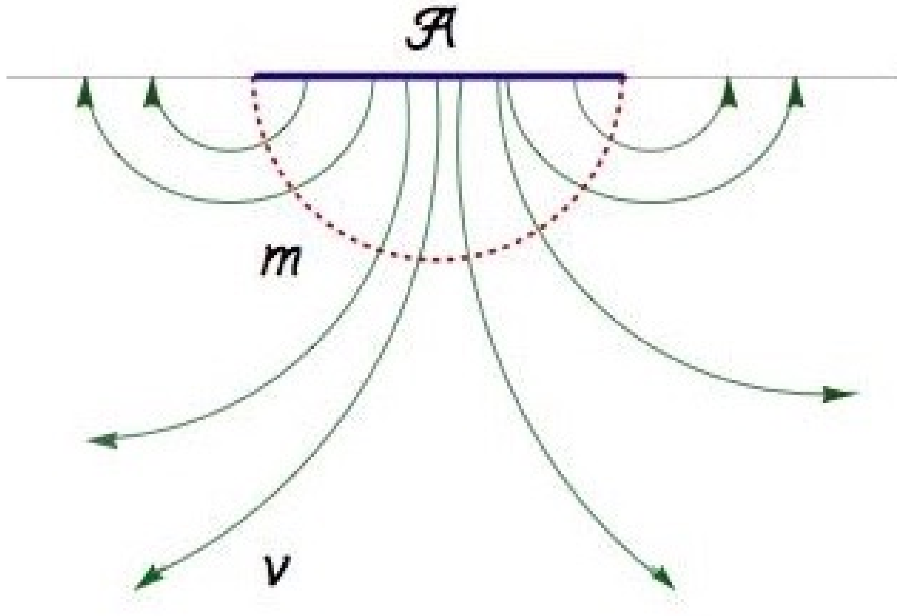
The LO (Local Operations) is

$$\rho \to \sum_{i,j} (A_i \otimes B_j).\rho.(A_i^\dagger \otimes B_j^\dagger)$$

where

$$\sum_i A_i^\dagger A = 1, \quad \sum_j B_j^\dagger B_j = 1$$

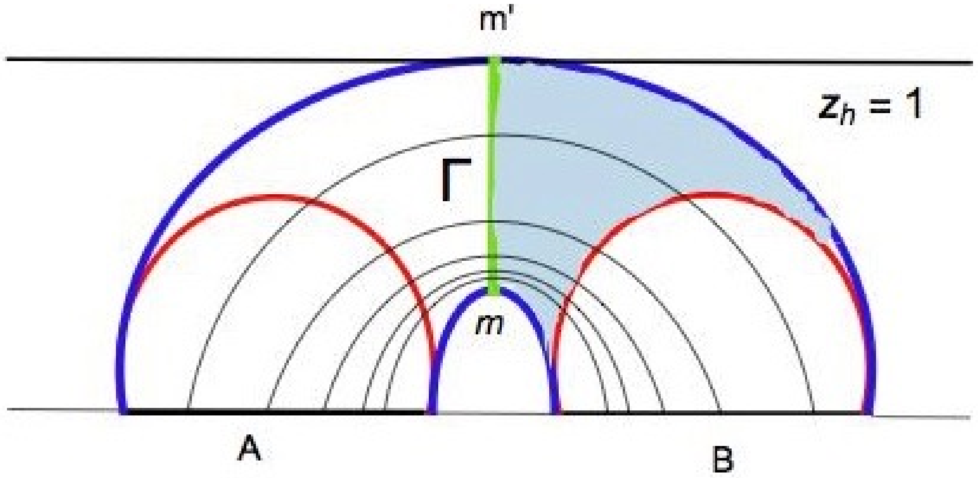
Operational and bit thread interpretations

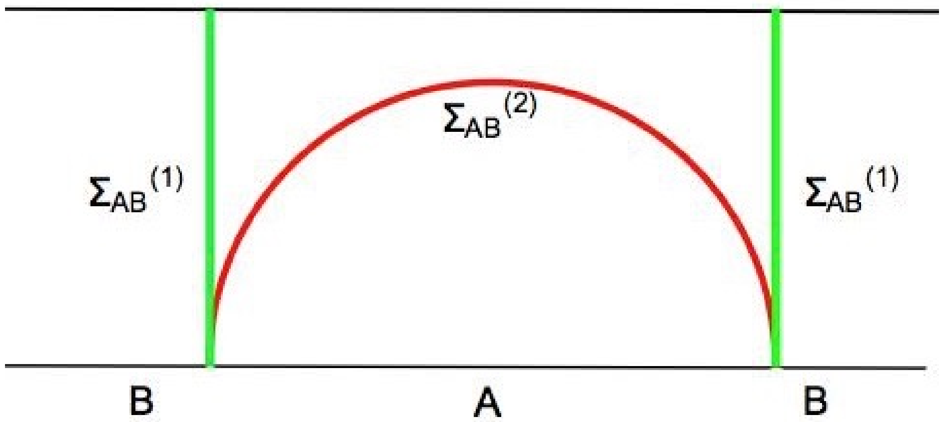


$$\mathcal{S}(\mathcal{A}) = \max_{\vec{v}} \int_{\mathcal{A}} \vec{v} \geq \int_{\mathcal{A}} \vec{v}.$$

Max-Flow, Min-Cut

Operational and bit thread interpretations





Thank You!